ON NONLINEAR PROBLEMS OF HEAT AND MASS TRANSFER

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A system of nonlinear equations for steady one-dimensional heat and mass transfer problems is considered. Analytical solutions are obtained for certain special cases.

In a previous paper (4) we have considered a nonlinear system of differential equations of steady-state heat and mass transfer, and we have presented the solutions for several special cases. In the present work we consider the more general system

$$\frac{d}{dx}\left\{a(u)\frac{du}{dx}+a(u)\delta(u,t)\frac{dt}{dx}\right\}=h(u),$$
$$\frac{d}{dx}\left\{\lambda(t)\frac{dt}{dx}\right\}=q(t), \tag{A}$$

where the function $\delta(u, t)$ is a function of either t or u only. We shall consider certain cases in which the nonlinear system (A) can be solved exactly by analytic methods.

The second equation in (A) can be rewritten in the form

$$\lambda(t)\frac{d^2t}{dx^2}+\lambda'(t)\left(\frac{dt}{dx}\right)^2-q(t)=0, \tag{B}$$

where

$$\lambda'(t)=\frac{d\lambda}{dt}.$$

Introducing the variables

$$\lambda_1(t)=\lambda'(t)/\lambda(t),\quad q_1(t)=-q(t)/\lambda(t),$$

we can transform (B) into

$$\frac{d^2t}{dx^2}+\lambda_1(t)\left(\frac{dt}{dx}\right)^2-q_1(t)=0. \tag{B'}$$

Equation (B') can be completely solved by analytic methods for a wide class of functions λ_1 and q_1 with boundary conditions of the first, second, or third kind. We shall demonstrate three possible methods of solution for this nonlinear equation.

1. Assuming $z(t) = dt/dx$, we can transform equation (B') into Bernoulli's equation,

$$\frac{dz}{dt}+\lambda_1(t)z=\frac{q_1(t)}{z},$$

for the unknown $z = z(t)$, which can be solved analytically for all continuous λ_1 and q_1. Thus equation (B') can be solved by two quadratures.

2. Using the substitution $v(t) = (dt/dx)^2$, we can reduce the nonlinear equation (B') to the linear equation

$$\frac{dv}{dt}+2\lambda_1(t)v=2q_1(t).$$

Integrating this equation, we obtain the function $v = v(t) + c_1$ and then find the function $t = t(x)$ from the equation

$$\int\frac{dt}{\sqrt{v(t)+c_1}}=x+c_2.$$

3. Multiplying (B) by $\pm 2\lambda(t)dt/dx$ (the plus sign corresponds to $\lambda > 0$, we obtain for the function $t = t(x)$ the equation

$$|\lambda(t)|^2\left(\frac{dt}{dx}\right)^2\mp 2\int|\lambda(t)|q(t)\,dt=c_1,$$

which can be integrated analytically.

The constants c_1 and c_2 are determined by the boundary conditions for t, which may be of the first, second, or third kind.

Thus the equation (B) or (B') can be completely solved by two quadratures for a wide class of functions $\lambda(t)$, $q(t)$, and $\lambda'(t)$.

For the sake of brevity we shall not present here the solutions of (B') which we have obtained for various forms of the functions $\lambda(t)$ and $q(t)$ with boundary conditions of the first, second, or third kind. Solving (B) or (B') by one of the above methods, we can obtain t as a function of the independent variable x.

In the following discussion we shall show that the integration of the nonlinear system (A) depends, in general, on the possibility of solving the first equation of this system.

In fact, assume $\delta = \delta(t)$. Then, solving the second equation, we can represent the derivative dt/dx by means of a known function $f(x)$, i.e. $dt/dx = f(x)$. The first equation of (A) then becomes

$$\frac{d}{dx}\left\{a(u)\frac{du}{dx}+a(u)\delta(t)f(x)\right\}=h(u). \tag{C}$$

We shall now consider those cases in which we can obtain an exact solution of the nonlinear equation (C), and hence a solution of the system (A), by analytic methods.

I. First consider the case when $\delta(t) = 1/\sqrt{v(t) + c_1}$ (2) or $\delta(t) = 1/f(x)$. The first equation of (A) then becomes

$$\frac{d^2u}{dx^2} + a_1(u)\left(\frac{du}{dx}\right)^2 + a_1(u)\frac{du}{dx} + h_1(u) = 0, \quad (1)$$

where

$$a_1(u) = \frac{1}{a(u)}\frac{da}{du}, \quad h_1(u) = -\frac{h(u)}{a(u)}.$$

The substitution $p(u) = du/dx$ reduces (1) to Abel's equation

$$p\frac{dp}{du} + a_1(u)p^2 + a_1(u)p + h_1(u) = 0, \quad (2)$$

which can be solved only in certain special cases.

The case $h(u) \equiv 0$ (and, consequently, $h_1(u) \equiv 0$) is of particular interest. In this case equation (2) becomes linear in p and can easily be integrated for a wide class of function $a_1(u)$. Thus, if $h \equiv 0$ the system (A) can be integrated to the end by analytic methods. This case has a practical application in problems of heat and mass transfer. Some solutions of this case were given (4).

II. Let now $\delta = \delta(t)$ be an arbitrary function and let $h(u) = au$, where $a(u) = a = \text{const}$. Solving the second equation in (A) we find $t = t(x)$. The product $\delta(t)dt/dx$ is then a known function $f(x)$. The first equation in (A) becomes, then,

$$\frac{d^2u}{dx^2} - u = -f'(x), \quad (3)$$

where $f'(x) = df/dx$. Equation (3) is the well known equation of forced oscillations, which can be integrated analytically to the end. Its general solution is

$$u = c_3\exp(x) + c_4\exp(-x) - \int_0^x f'(t)\,\mathrm{sh}(x-t)\,dt.$$

Taking account of the boundary conditions (of the first, second, or third kind), we obtain the constants c_3 and c_4. Thus, the nonlinear system (A) can be completely solved also in this case.

III. Let $a(u) = a = \text{const}$, $\delta(u) = u$, $h(u) = 0$ and let $dt/dx = f(x)$ be a known function, obtained by differentiating the solution of the second equation of the system (A). The first equation of the system then becomes

$$\frac{d}{dx}\left\{\frac{du}{dx} + uf(x)\right\} = 0. \quad (4)$$

or

$$\frac{d^2u}{dx^2} + f(x)\frac{du}{dx} + f'(x)u = 0. \quad (4')$$

The solution of this equation is

$$u = e^{-F}[C_3 + C_4\int e^F dx],$$

where $F(x) = \int f(x)dx$. The constants c_3 and c_4 are then found from the boundary conditions for $u = u(x)$, yielding the final solution of (A).

IV. System (A) can be solved also in a case more general than the last one. Let, in contrast to Case III, $h(u) = a[q(x) - bu]$, where $q(x)$ is a known function and b is a constant. Instead of equation (4) we now have

$$\frac{d^2u}{dx^2} + f(x)\frac{du}{dx} + [f'(x) + b]u = q(x). \quad (5)$$

This is the well known equation of super-regeneration of a receiver (2). The general solution of (5), with the constants c_3 and c_4 determined from the boundary condition, constitutes an exact analytic solution of the nonlinear system under consideration.

In conclusion we shall present another case, even more complex than the last one, in which system (A) can be solved by quadratures.

V. Let $a(u) = a = \text{const}$, $\delta(u) = u$, $dt/dx = f(x)$ and $h(u) = a/u$. The first equation in (A) becomes now

$$\frac{d^2u}{dx^2} + f(x)\frac{du}{dx} + f'(x)u = \frac{1}{u}. \quad (6)$$

This equation has been studied in detail by Leko [3]. Its general solution is

$$\int\exp[\int f(x)dx]dx = c_3\int\exp[\int\varepsilon d\varepsilon]d\varepsilon + c_4,$$
$$u = c_3\exp(\int\varepsilon d\varepsilon)\exp[-\int f(x)dx],$$

where ε is a parameter and c_3, c_4 are arbitrary constants determined as before by the boundary conditions for $u = u(x)$. The parameter ε is introduced by the auxiliary equation

$$\frac{d^2x}{d\varepsilon^2} + f(\varepsilon)\left(\frac{dx}{d\varepsilon}\right)^2 - \varepsilon\frac{dx}{d\varepsilon} = 0,$$

which is used in the integration of (6).

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